

# THE SURREAL NUMBERS AS A UNIVERSAL $H$ -FIELD

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ABSTRACT. We show that the natural embedding of the differential field of transseries into Conway's field of surreal numbers with the Berarducci-Mantova derivation is an elementary embedding. We also prove that any Hardy field embeds into the field of surreals with the Berarducci-Mantova derivation.

## INTRODUCTION

Berarducci and Mantova [BM, Theorem B] have recently constructed a derivation (denoted by  $\partial_{\text{BM}}$  below) on Conway's ordered field  $\mathbf{No}$  of surreal numbers that makes the latter a Liouville closed  $H$ -field with constant field  $\mathbb{R}$ . The standard example of such an object is the ordered differential field  $\mathbb{T}$  of transseries, and the question arises whether  $\mathbf{No}$  with  $\partial_{\text{BM}}$  is elementarily equivalent to  $\mathbb{T}$ . Below we give a positive answer in a stronger form: Theorem 1. Throughout this paper we consider  $\mathbf{No}$  as a differential field with derivation  $\partial_{\text{BM}}$ .

Both  $\mathbf{No}$  and  $\mathbb{T}$  are also exponential fields; the exponential function  $\exp$  on  $\mathbf{No}$  is defined in Gonshor [G]. We refer to [ADH, Appendix A] for the precise construction of  $\mathbb{T}$ , but the “generating element”  $x$  of  $\mathbb{T}$  there will be denoted by  $x_{\mathbb{T}}$  here, since we prefer to have  $x$  range here over arbitrary surreal numbers. It is folklore (but see Section 5 for a proof) that there is a unique embedding  $\iota: \mathbb{T} \rightarrow \mathbf{No}$  of ordered exponential fields with  $\iota(x_{\mathbb{T}}) = \omega$  that is the identity on  $\mathbb{R}$  and respects infinite sums. It follows easily from Wilkie's theorem [W] and other known facts that  $\iota$  is an elementary embedding of ordered exponential fields; see Section 5 for details. The analogue for the derivation instead of the exponentiation requires more effort:

**Theorem 1.** *The mapping  $\iota: \mathbb{T} \rightarrow \mathbf{No}$  is an elementary embedding of ordered differential fields.*

This answers a question posed in [BM]. The main tools for proving this result come from [ADH, Theorems 15.0.1 and 16.0.1]. These tools enable us to reduce the proof of Theorem 1 to exhibiting  $\mathbf{No}$  as a directed union of subfields  $\mathbb{R}[[\omega^{\Gamma}]]$  that are closed under  $\partial_{\text{BM}}$  and where  $\Gamma$  is an ordered additive subgroup of  $\mathbf{No}$  having a smallest nontrivial archimedean class; exhibiting  $\mathbf{No}$  as such a directed union makes up an important part of our paper. (As a byproduct we get a new proof that  $\partial_{\text{BM}}(\mathbf{No}) = \mathbf{No}$ .) We use the same kind of reduction to obtain:

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**Theorem 2.** *The surreals of countable length form a subfield of  $\mathbf{No}$  closed under  $\partial_{\text{BM}}$ . As a differential subfield of  $\mathbf{No}$  it is an elementary submodel of  $\mathbf{No}$ .*

This also uses a result of Esterle [E] and its consequence that for any countable ordinal  $\alpha$ , any well-ordered set of surreals of length  $< \alpha$  is countable: Lemma 25.

Finally, we establish an embedding result for  $H$ -fields:

**Theorem 3.** *Every  $H$ -field with small derivation and constant field  $\mathbb{R}$  can be embedded over  $\mathbb{R}$  as an ordered differential field into  $\mathbf{No}$ .*

Thus every Hardy field extending  $\mathbb{R}$  embeds over  $\mathbb{R}$  as an ordered differential field into  $\mathbf{No}$ . Despite these excellent properties of  $\partial_{\text{BM}}$ , Schmeling's thesis [S] gives us reason to believe that  $\partial_{\text{BM}}$  is not yet the “best” derivation on  $\mathbf{No}$ . We expect to address this issue in later papers.

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## 1. PRELIMINARIES

Here we fix notation and terminology and summarize the results from [ADH, BM] that we need as background material and as tools in our proofs.

### Notations and terminology.

Below,  $m, n$  range over  $\mathbb{N} = \{0, 1, 2, \dots\}$ , and  $\alpha, \beta$  and  $\mu, \nu$  range over ordinals. (The letter  $\lambda$  will serve another purpose, as in [BM].)

As in [G], a *surreal number* is by definition a function  $a: \mu \rightarrow \{-, +\}$  on an ordinal  $\mu = \{\alpha: \alpha < \mu\}$ . For such  $a$  we let  $l(a) := \mu$  be the *length* of  $a$ . From now on we let  $a, b, x, y$  be surreal numbers. The class  $\mathbf{No}$  of surreal numbers carries a canonical linear ordering  $<$ :  $a < b$  iff  $a$  is lexicographically less than  $b$ , where by convention we set  $a(\mu) := 0$  for  $\mu \geq l(a)$  and linearly order  $\{-, 0, +\}$  by  $- < 0 < +$ . We also have the canonical partial ordering  $<_s$  on  $\mathbf{No}$  given by:  $a <_s b$  (“ $a$  is simpler than  $b$ ”) iff  $a$  is a proper initial segment of  $b$ , that is,  $l(a) < l(b)$ , and  $a|_\mu = b|_\mu$  for  $\mu := l(a)$ . For sets  $A, B \subseteq \mathbf{No}$  with  $A < B$  (that is,  $a < b$  for all  $a \in A$  and  $b \in B$ ) we let  $x = A|B$  mean that  $x$  is the simplest surreal with  $A < x < B$ , as in [G] and [BM]. We also use the terms “canonical representation” and “monomial representation” (of a surreal number) as in [BM].

The ordinal  $\alpha$  is identified with the surreal  $a: \alpha \rightarrow \{-, +\}$  with  $a(\beta) = +$  for all  $\beta < \alpha$ . A useful fact is the equivalence  $\alpha < x \iff \alpha \dot{+} 1 \leq_s x$ , where  $\alpha \dot{+} 1$  is the successor ordinal to  $\alpha$ . The subclass of  $\mathbf{No}$  consisting of the ordinals is denoted by  $\mathbf{On}$ . A set  $S \subseteq \mathbf{No}$  is said to be *initial* if  $x \in S$  whenever  $x <_s y \in S$ . As in [DE] we set  $\mathbf{No}(\alpha) = \{x: l(x) < \alpha\}$ , an initial subset of  $\mathbf{No}$ .

We refer to [G] or [BM] for the inductive definitions of the binary operations of addition and multiplication on  $\mathbf{No}$  that make  $\mathbf{No}$  into a real closed field, with the ordinal 0 as its zero element and the ordinal 1 as its multiplicative identity. The field ordering of this real closed field is the above lexicographic linear ordering  $<$ . This field  $\mathbf{No}$  contains  $\mathbb{R}$  as an initial subfield in the way specified in [G]. The field sum  $\alpha + n$  equals the ordinal sum  $\alpha \dot{+} n$ . Each initial set  $\mathbf{No}(\omega^\alpha)$  underlies an additive subgroup of  $\mathbf{No}$ ; see [DE].

Let  $\Gamma$  be an (additively written) ordered abelian group. Then we set

$$\Gamma^> := \{\gamma \in \Gamma: \gamma > 0\}.$$

We use this notation also for the underlying additive groups of  $\mathbf{No}$  and  $\mathbb{R}$ , so  $\mathbf{No}^> = \{a: a > 0\}$ , and  $\mathbb{R}^> := \{r \in \mathbb{R} : r > 0\}$ . For  $\gamma \in \Gamma$  we define

$$[\gamma] := \{\delta \in \Gamma : |\delta| \leq n|\gamma| \text{ and } |\gamma| \leq n|\delta| \text{ for some } n \geq 1\},$$

the *archimedean class* of  $\gamma$  (in  $\Gamma$ ). The archimedean classes of elements of  $\Gamma$  partition the set  $\Gamma$ , and we totally order this set of archimedean classes by

$$[\gamma_1] < [\gamma_2] \quad :\Longleftrightarrow \quad n|\gamma_1| < |\gamma_2| \text{ for all } n \geq 1.$$

Thus the least archimedean class is  $[0] = \{0\}$ , the *trivial* archimedean class.

The convex hull of  $\mathbb{R}$  in  $\mathbf{No}$  is a valuation ring  $V$  of the field  $\mathbf{No}$ . We consider  $\mathbf{No}$  accordingly as a *valued* field whose (Krull) valuation  $v$  has  $V$  as its valuation ring. For any (Krull) valued field  $K$  with valuation  $v$  and elements  $f, g \in K$  we let  $f \leq g$ ,  $f \prec g$ ,  $f \asymp g$ ,  $f \sim g$  abbreviate  $v(f) \geq v(g)$ ,  $v(f) > v(g)$ ,  $v(f) = v(g)$ , and  $v(f - g) > v f$ . (See [ADH, Section 3.1].) We shall use these notations in particular for the valued field  $\mathbf{No}$ .

### The omega map, the Conway normal form, and summability.

We assume familiarity with Conway's omega map  $x \mapsto \omega^x: \mathbf{No} \rightarrow \mathbf{No}^>$ . Recall that  $\omega^x$  is the simplest positive element in its archimedean class; so  $\omega^x \prec \omega^y$  whenever  $x < y$ . See [G] for details, including the proof that each  $a$  has a unique representation

$$a = \sum_x a_x \omega^x \quad (\text{the Conway normal form of } a)$$

with real coefficients  $a_x$  such that  $E(a) := \{x: a_x \neq 0\}$  is a subset of  $\mathbf{No}$  (not just a subclass) and is reverse well-ordered. This will be the meaning of  $E(a)$  and  $a_x$  throughout. The *leading monomial* of  $a$  is  $\omega^x$  with  $x = \max E(a)$ , for  $a \neq 0$ . The *terms* of  $a$  are the  $a_x \omega^x$  with  $a_x \neq 0$ . The omega map extends the usual ordinal exponentiation  $\alpha \mapsto \omega^\alpha$ . Given any set  $S \subseteq \mathbf{No}$  we let  $\mathbb{R}[[\omega^S]]$  denote the additive subgroup of  $\mathbf{No}$  consisting of the surreals  $a$  with  $E(a) \subseteq S$ .

Let  $(a_i)_{i \in I}$  be a family of surreals; this includes  $I$  being a set. We say that  $(a_i)$  is *summable* (or that  $\sum_i a_i$  exists) if  $\bigcup_i E(a_i)$  is reverse well-ordered, and for each  $x$  there are only finitely many  $i \in I$  with  $x \in E(a_i)$ ; in that case we set  $\sum_i a_i := \sum_x (\sum_i a_{i,x}) \omega^x$ . If  $S$  is a subset of  $\mathbf{No}$ , then for any summable family  $(a_i)$  in  $\mathbb{R}[[\omega^S]]$  we have  $\sum_i a_i \in \mathbb{R}[[\omega^S]]$ .

As in [BM], we let  $\mathfrak{M}$  denote the class of *monomials*  $\omega^x$ ; so  $\mathfrak{M}$  is a multiplicative subgroup of  $\mathbf{No}^\times$ . The Conway normal form allows us to consider any surreal number  $a$  as a *generalized series*

$$a = \sum_{\mathfrak{m} \in \mathfrak{M}} a_{\mathfrak{m}} \mathfrak{m}$$

with coefficients  $a_{\mathfrak{m}} \in \mathbb{R}$ , monomials  $\mathfrak{m} \in \mathfrak{M}$ , and reverse well-ordered *support*  $\text{supp } a := \{\mathfrak{m} \in \mathfrak{M} : a_{\mathfrak{m}} \neq 0\} = \omega^{E(a)}$ . This makes the above notion of summability for surreal numbers coincide with the corresponding notion for generalized series from [S, Section 1.5].

Next,  $\mathbb{J} := \{a: E(a) \subseteq \mathbf{No}^>\}$  is the class of *purely infinite* surreals, an additive subgroup of  $\mathbf{No}$  that is moreover closed under multiplication. Thus  $\mathfrak{M} \cap \mathbb{J} = \mathfrak{M}^{\prec 1}$ , and  $\mathbf{No} = \mathbb{J} \oplus \mathbb{R} \oplus \mathbf{No}^{\prec 1}$ .

### Exponentiation, and the functions $g$ and $h$ .

Gonshor [G] gave an inductive definition of the exponential function  $\exp: \mathbf{No} \rightarrow \mathbf{No}^>$ , and established its basic properties. These include  $\exp$  being an order-preserving isomorphism from the additive group of  $\mathbf{No}$  onto its multiplicative group of positive elements. The inverse of  $\exp$  is of course denoted by  $\log: \mathbf{No}^> \rightarrow \mathbf{No}$ . The  $n$ th iterate of the map  $\exp: \mathbf{No} \rightarrow \mathbf{No}$  is denoted by  $\exp_n$ , so  $\exp_0$  is the identity map on  $\mathbf{No}$ , and  $\exp_1(x) = \exp(x)$ . Also  $e^x := \exp(x)$ . The logarithmic map  $\log$  maps  $\mathbf{No}^{>\mathbb{N}}$  into itself; the  $n$ th iterate of the restriction of  $\log$  to a map  $\mathbf{No}^{>\mathbb{N}} \rightarrow \mathbf{No}^{>\mathbb{N}}$  is denoted by  $\log_n$ , so  $\log_0$  is the identity map on  $\mathbf{No}^{>\mathbb{N}}$  and  $\log_1(x) = \log(x)$  for  $x > \mathbb{N}$ .

The exponential map  $\exp$  and the omega-map  $x \mapsto \omega^x$  are related by the order preserving bijection  $g: \mathbf{No}^> \rightarrow \mathbf{No}$ , which satisfies

$$\exp(\omega^x) = \omega^{\omega^{g(x)}} \quad \text{for all } x > 0.$$

We have  $g(n) = n$  for all  $n$ . More generally, Theorem 10.14 in [G] says that  $g(\alpha) = \alpha$  unless  $\epsilon \leq \alpha < \epsilon + \omega$  for some  $\epsilon$ -number, in which case  $g(\alpha) = \alpha + 1$ . (An  $\epsilon$ -number is an ordinal  $\epsilon$  such that  $\omega^\epsilon = \epsilon$ .) We shall need  $g(x)$  mainly in the other extreme case where  $x$  has the form  $\omega^{-\alpha}$ . Here Theorem 10.15 in [G] gives  $g(\omega^{-\alpha}) = -\alpha + 1$ .

We also use the inverse  $h: \mathbf{No} \rightarrow \mathbf{No}^>$  of  $g$ . Note that

$$\omega^{\omega^y} = \exp(\omega^{h(y)}) \quad \text{for all } y.$$

The result above for  $g(\omega^{-\alpha})$  yields  $h(-\alpha + 1) = \omega^{-\alpha}$ , from which we get

$$\log \omega^{\omega^{-\alpha+1}} = \omega^{\omega^{-\alpha}}.$$

Applying this to the ordinal  $\alpha + 1$  instead of  $\alpha$  we get

$$\log \omega^{\omega^{-\alpha}} = \omega^{\omega^{-(\alpha+1)}}.$$

From [G] we have  $\exp(\mathbb{J}) = \mathfrak{M}$ . Thus besides the Conway normal form and the series representation, any surreal number  $a$  also has a unique representation

$$a = \sum_{j \in \mathbb{J}} a_j e^j \quad (\text{exponential normal form of } a)$$

with real coefficients  $a_j$  and reverse well-ordered  $\{j \in \mathbb{J} : a_j \neq 0\}$ ; this is also called the *Ressayre form of  $a$* . For nonzero  $a$  with leading monomial  $e^b$ ,  $b \in \mathbb{J}$ , we set  $\ell(a) := b$ . Then  $-\ell: \mathbf{No}^\times \rightarrow \mathbb{J}$  is a (Krull) valuation on the field  $\mathbf{No}$ , and

$$\{a: -\ell(a) \geq 0\} = \{a: |a| \leq r \text{ for some } r \in \mathbb{R}^{\geq 0}\} = V,$$

so we may consider  $-\ell$  as the valuation of our valued field  $\mathbf{No}$ . Important in [BM] is also the class  $\mathfrak{A}$  of *log-atomic* surreals, consisting of the  $a > \mathbb{N}$  all whose iterated logarithms  $\log_n a$  lie in  $\mathfrak{M}$ . We have  $\mathfrak{A} \subseteq \mathfrak{M}^{>1}$  and  $\exp(\mathfrak{A}) = \log(\mathfrak{A}) = \mathfrak{A}$ . It follows from  $\mathfrak{A} \subseteq \mathfrak{M}$  that if  $x, y \in \mathfrak{A}$  and  $x < y$ , then  $x \prec y$ . (In [BM] the class of log-atomic surreals is denoted by  $\mathbb{L}$ , but this notation conflicts with ours in other papers.)

### Surreal derivations..

We summarize here some results from [BM] as needed, and add a few remarks. A *surreal derivation* is a derivation  $\partial$  on the field  $\mathbf{No}$  such that

$$(SD1). \quad \{a: \partial(a) = 0\} = \mathbb{R};$$

$$(SD2). \quad \partial(a) > 0 \text{ for all } a > \mathbb{R};$$

(SD3).  $\partial(\exp(a)) = \partial(a) \exp(a)$  for all  $a$ ;

(SD4). for any summable family  $(a_i)$  of surreals, the family  $(\partial(a_i))$  is also summable, and  $\partial(\sum_i a_i) = \sum_i \partial(a_i)$ .

The ordered field  $\mathbf{No}$  equipped with any surreal derivation is an  $H$ -field; this doesn't need (SD3) or (SD4). The particular derivation  $\partial_{\text{BM}}$  is surreal, maps  $\mathfrak{A}$  into  $\mathfrak{M}$ , and is obtained in [BM] as a special case of a rather general construction. Before we get to that, we mention Proposition 6.5 and Theorem 6.32 from that paper:

(BM1). If  $\partial$  is a surreal derivation, then for all  $x, y > \mathbb{N}$  with  $x - y > \mathbb{N}$  we have

$$\log \partial(x) - \log \partial(y) \prec x - y.$$

(BM2). Any map  $D: \mathfrak{A} \rightarrow \mathbb{R}^{>\mathfrak{M}}$  such that for all  $x, y \in \mathfrak{A}$ ,

$$D(\exp x) = D(x) \exp x, \quad \log D(x) - \log D(y) \prec \max(x, y),$$

extends to a surreal derivation.

Thus (BM2) is a partial converse to (BM1), although the condition in (BM2) that  $D$  takes only values in  $\mathbb{R}^{>\mathfrak{M}}$  seems a rather severe restriction. We define a *pre-derivation* to be a map  $D: \mathfrak{A} \rightarrow \mathbb{R}^{>\mathfrak{M}}$  as in (BM2). Note that if  $D$  is a pre-derivation, then

$$D(a) = \left( \prod_{m < n} \log_m a \right) \cdot D(\log_n a) \quad \text{for all } a \in \mathfrak{A} \text{ and all } n. \quad (1)$$

A pre-derivation  $D$  actually extends canonically to a surreal derivation  $\partial_D$ . To define  $\partial_D$  in terms of  $D$  we rely on the notion of *path derivatives*, introduced in [vdH:phd], further developed in [S], and adapted to the surreal setting in [BM]. A *path* is a function  $P: \mathbb{N} \rightarrow \mathbb{R}^{\times \mathfrak{M}}$  such that  $P(n+1)$  is a term of  $\ell(P(n))$ , for all  $n$ . Given  $x$ , the paths  $P$  such that  $P(0)$  is a term of  $x$  are the elements of a set  $\mathcal{P}(x)$ . For  $x \in \mathfrak{A}$  there is a unique path  $P \in \mathcal{P}(x)$ ; it is given by  $P(n) = \log_n x$ . Thus if  $P$  is a path and  $P(m) \in \mathfrak{A}$ , then  $P(n) = \log_{n-m} P(m)$  for all  $n \geq m$ , so  $P(n) \in \mathfrak{A}$  for all  $n \geq m$ .

Let  $D$  be a pre-derivation. The *path derivative*  $\partial_D(P) \in \mathbb{R}^{\mathfrak{M}}$  for a path  $P$  is defined as follows, with (1) guaranteeing independence of  $n$  in (1):

1. if  $P(n) \in \mathfrak{A}$ , then  $\partial_D(P) := (\prod_{m < n} P(m)) \cdot D(P(n))$ ;
2. if  $P(n) \notin \mathfrak{A}$  for all  $n$ , then  $\partial_D(P) := 0$ .

The rationale behind path derivatives is the following proposition:

(BM3). For each  $a$  the family  $(\partial_D(P))_{P \in \mathcal{P}(a)}$  is summable.

This result is stated in [BM, Proposition 6.20] only for one particular pre-derivation, but, as the authors mention, the proof extends to any pre-derivation. In view of (BM3) we can now define  $\partial_D: \mathbf{No} \rightarrow \mathbf{No}$  by

$$\partial_D(a) := \sum_{P \in \mathcal{P}(a)} \partial_D(P).$$

It follows from (1) that  $\partial_D$  extends  $D$ , and the arguments in [BM, Section 6] show that  $\partial_D$  is a surreal derivation.

**Results from [ADH].**

To state the relevant facts, we recall from [AD] or [ADH] that an  $H$ -field is by definition an ordered differential field  $K$  with derivation  $\partial$  and constant field  $C = \{f \in K : \partial(f) = 0\}$  such that:

- (H1).  $\partial(f) > 0$  for all  $f \in K$  with  $f > C$ ;
- (H2).  $\mathcal{O} = C + \mathfrak{o}$ , where  $\mathcal{O}$  is the convex hull of  $C$  in  $K$ , and  $\mathfrak{o}$  is the maximal ideal of the valuation ring  $\mathcal{O}$ .

Let  $K$  be an  $H$ -field, and let  $\mathcal{O}$  and  $\mathfrak{o}$  be as in (H2). Thus  $K$  is a valued field with valuation ring  $\mathcal{O}$ . We consider  $K$  in the natural way as an  $\mathcal{L}$ -structure, where

$$\mathcal{L} := \{0, 1, +, -, \times, \partial, \leq, \preceq\}$$

is the language of ordered valued differential fields; in particular,

$$f \preceq g \iff f \in \mathcal{O}g \iff |f| \leq c|g| \text{ for some } c \geq 0 \text{ in } C.$$

Given  $f \in K$  we also write  $f'$  instead of  $\partial(f)$ , and we set  $f^\dagger := f'/f$  for  $f \neq 0$ , so  $(fg)^\dagger = f^\dagger + g^\dagger$  and  $(1/f)^\dagger = -f^\dagger$  for  $f, g \in K^\times$ . A useful subset of the value group  $\Gamma := v(K^\times)$  of the valued field  $K$  is

$$\Psi := \Psi_K := \{v(f^\dagger) : f \in K^\times, f \neq 1\} = \{v(f^\dagger) : f \in K, f > C\}.$$

As in [ADH] we call  $K$  *grounded* if  $\Psi$  has a largest element. For the convenience of the reader we include a proof of the following wellknown fact.

**Lemma 4.** *Assume  $K$  has constant field  $C = \mathbb{R}$ . Then  $K$  is grounded iff  $\Gamma$  has a smallest nontrivial archimedean class.*

**Proof.** Let  $f, g \in K$ ,  $f, g > C$ . Suppose the archimedean class  $[v(f)] = [v(1/f)]$  of  $v(f)$  is greater than  $[v(g)]$ . This means:  $v(f) < n v(g) = v(g^n) < 0$  for all  $n \geq 1$ . Hence  $f^\dagger > (g^n)^\dagger = n g^\dagger > 0$  for all  $n \geq 1$ , by [AD, Lemma 1.4], so  $v(f^\dagger) < v(g^\dagger)$ . A similar argument (which doesn't need  $C = \mathbb{R}$ ) shows that if  $[v(f)] = [v(g)]$ , then  $v(f^\dagger) = v(g^\dagger)$ . Thus we have an order-reversing bijection  $[v(f)] \mapsto v(f^\dagger)$  ( $f \in K$ ,  $f > C$ ) from the set of nontrivial archimedean classes of  $\Gamma$  onto  $\Psi$ .  $\square$

An  $H$ -subfield of  $K$  is by definition an ordered differential subfield of  $K$  that is an  $H$ -field. In [ADH] we axiomatized the elementary (= first-order) theory of the  $H$ -field  $\mathbb{T}$  of transseries. This (complete) theory is called  $T_{\text{small}}^{\text{nl}}$  there and its models are exactly the  $H$ -fields  $K$  satisfying the following (first-order) conditions:

1. the derivation of  $K$  is small, that is,  $\partial\mathcal{O} \subseteq \mathfrak{o}$ ;
2.  $K$  is Liouville closed;
3.  $K$  is  $\omega$ -free;
4.  $K$  is newtonian.

(An  $H$ -field  $K$  is said to be *Liouville closed* if it is real closed and for all  $f \in K$  there exists  $g \in K$  with  $g' = f$  and an  $h \in K^\times$  such that  $h^\dagger = f$ ; for the definition of “ $\omega$ -free” and “newtonian” we refer to the Introduction of [ADH].) Dropping the smallness axiom (1), we get the incomplete but model complete theory  $T^{\text{nl}}$ ; see [ADH, Chapter 16]. The  $H$ -field  $\mathbb{T}$  satisfies (3) and (4) by [ADH, Corollary 11.7.15 and Theorem 15.0.1], which for an arbitrary  $H$ -field  $K$  amount to the following:

*If  $\partial K = K$  and  $K$  is a directed union of spherically complete grounded  $H$ -subfields, then  $K$  is  $\omega$ -free and newtonian.*

The condition  $\partial K = K$  is automatically satisfied if  $K$  is a directed union of spherically complete grounded  $H$ -subfields  $E$  such that for some  $\phi \in E$  we have  $v(\phi) = \max \Psi_E$  and  $\phi \in \partial K$ , by [ADH, Corollary 15.2.4].

## 2. INFINITE PRODUCTS AND LOG-ATOMIC SURREALS

The pre-derivation  $D$  in [BM] with  $\partial_D = \partial_{\text{BM}}$  is defined by a certain identity. Towards the end of this section we give this identity a more suggestive form, which we found useful. But we begin with some remarks on  $\epsilon$ -numbers, which play an important role in the next sections.

**Remarks on  $\epsilon$ -numbers.**

Throughout this paper  $\epsilon$  will denote an  $\epsilon$ -number, that is,  $\epsilon$  is an ordinal such that  $\omega^\epsilon = \epsilon$ .

**Lemma 5.** *For any  $\alpha$  there is a least  $\epsilon$ -number  $\epsilon(\alpha) \geq \alpha$ . Moreover, if  $\alpha$  is infinite, then  $\text{card}(\epsilon(\alpha)) = \text{card}(\alpha)$ .*

**Proof.** The recursion defining  $\omega^\alpha$  as a function of  $\alpha$  easily yields that this function is strictly increasing, with  $\omega^\alpha \geq \alpha$ ,  $\text{card}(\omega^\alpha) = \max(\aleph_0, \text{card}(\alpha))$ , and thus  $\text{card}(\omega^\alpha) = \text{card}(\alpha)$  if  $\alpha$  is infinite. Now define  $\alpha_n$  as a function of  $n$  by the recursion  $\alpha_0 = \alpha$  and  $\alpha_{n+1} = \omega^{\alpha_n}$ . Then  $\sup_n \alpha_n$  is clearly the least  $\epsilon$ -number  $\geq \alpha$ , and it has the same cardinality as  $\alpha$  if the latter is infinite.  $\square$

If  $\kappa$  is an uncountable cardinal, then by the remarks in the proof above we have  $\omega^\alpha < \kappa$  for all  $\alpha < \kappa$ . Thus uncountable cardinals are  $\epsilon$ -numbers. The least  $\epsilon$ -number is denoted by  $\epsilon_0$ , as usual, so  $\epsilon_0 = \sup_n \omega_n$  where the  $\omega_n$  are defined by the recursion  $\omega_0 = \omega$  and  $\omega_{n+1} = \omega^{\omega_n}$ .

**Infinite products of monomials.**

Recall that  $\mathfrak{M}$  is the multiplicative group of monomials  $\omega^a$ . For a family  $(\mathfrak{m}_i)$  in  $\mathfrak{M}$  we say that  $\prod_i \mathfrak{m}_i$  exists if  $\sum_i a_i$  exists, with  $\mathfrak{m}_i = \omega^{a_i}$  for all  $i$ , and in that case, we set

$$\prod_i \mathfrak{m}_i := \omega^{\sum_i a_i} \in \mathfrak{M}.$$

The rules for manipulating these infinite products are easy consequences of those for infinite sums, and we shall freely use them below. Note in particular that if  $(\mathfrak{m}_i)$  is a family in  $\mathfrak{M}$  and  $\prod_i \mathfrak{m}_i$  exists, then  $\prod_i \mathfrak{m}_i^{-1}$  exists and equals  $(\prod_i \mathfrak{m}_i)^{-1}$ .

In our definition of infinite products we could have represented monomials as exponentials of elements in  $\mathbb{J}$  instead of as powers of  $\omega$ . Indeed, the equivalence between these options follows from the next two lemmas:

**Lemma 6.** *Let  $(a_i)$  be a summable family in  $\mathbb{J}$ . Then  $\prod_i \exp(a_i)$  exists, and*

$$\exp\left(\sum_i a_i\right) = \prod_i \exp(a_i).$$

**Proof.** We have  $a_i = \sum_{x>0} a_{i,x} \omega^x$ , so by [G, Theorem 10.13],

$$\exp(a_i) = \omega^{b_i}, \quad b_i := \sum_{x>0} a_{i,x} \omega^{g(x)},$$

so  $E(b_i) = g(E(a_i))$ . Since  $\sum_i a_i$  exists, so does  $\sum_i b_i$ , and hence  $\prod_i \exp(a_i) = \prod_i \omega^{b_i}$  exists, and  $\prod_i \exp(a_i) = \omega^{\sum_i b_i}$ . Moreover, with  $\sum_i a_i = \sum_{x>0} a_x \omega^x$ , we have  $\sum_i b_i = \sum_{x>0} a_x \omega^{g(x)}$ . Hence again by [G, Theorem 10.13],

$$\prod_i \exp(a_i) = \omega^{\sum_{x>0} a_x \omega^{g(x)}} = \exp\left(\sum_{x>0} a_x \omega^x\right) = \exp\left(\sum_i a_i\right),$$

as claimed.  $\square$

**Lemma 7.** *Let  $(\mathfrak{m}_i)$  be a family in  $\mathfrak{M}$  such that  $\prod_i \mathfrak{m}_i$  exists. Then  $\sum_i \log \mathfrak{m}_i$  exists, and  $\log \prod_i \mathfrak{m}_i = \sum_i \log \mathfrak{m}_i$ .*

**Proof.** We have  $\mathfrak{m}_i = \exp(a_i)$  with  $a_i \in \mathbb{J}$ , so  $a_i = \sum_{x>0} a_{i,x} \omega^x$ , hence

$$\mathfrak{m}_i = \omega^{b_i}, \quad b_i := \sum_{x>0} a_{i,x} \omega^{g(x)}$$

by [G, Theorem 10.13]. Since the product  $\prod_i \mathfrak{m}_i$  exists, so does  $\sum_i b_i$ , and therefore  $\sum_i a_i = \sum_i \log \mathfrak{m}_i$  exists. Moreover, and again by [G, Theorem 10.13],

$$\prod_i \mathfrak{m}_i = \omega^{\sum_i b_i} = \omega^{\sum_{x>0} a_x \omega^{g(x)}} = \exp\left(\sum_{x>0} a_x \omega^x\right), \quad a_x := \sum_i a_{i,x},$$

and so  $\log \prod_i \mathfrak{m}_i = \sum_{x>0} a_x \omega^x = \sum_i a_i$ .  $\square$

### Log-atomic surreals.

Recall that  $\mathfrak{A} \subseteq \mathfrak{M}^{>1}$  is the class of log-atomic surreals. See [BM, Sections 1, 5] for the order-preserving bijection  $x \mapsto \lambda_x: \mathbf{No} \rightarrow \mathfrak{A}$  and for the fact that  $\lambda_x \leq_s \lambda_y$  iff  $x \leq_s y$ . It follows from  $\exp(\omega^x) = \omega^{\omega^{g(x)}}$  that  $\mathfrak{A} \subseteq \omega^{\mathfrak{M}}$ . Thus for any well-ordered index set  $I$  and strictly decreasing map  $i \mapsto \lambda_i: I \rightarrow \mathfrak{A}$  the product  $\prod_i \lambda_i$  exists. We shall use Proposition 10 and Corollary 13 below to define the pre-derivation  $\partial_{\text{BM}}|_{\mathfrak{A}}$ .

**Lemma 8.** *Let  $\mathfrak{m} = A|B$  be a monomial representation with  $\mathfrak{m} \succ 1$ . Then*

$$\exp(\mathfrak{m}) = (\mathfrak{m}^{\mathbb{N}} \cup \exp(A)) | \exp(B).$$

**Proof.** For  $\mathfrak{m}' < \mathfrak{m}$  with  $\mathfrak{m}' <_s \mathfrak{m}$  we have  $\mathfrak{m}' \leq a$  for some  $a \in A$  (since  $A < \mathfrak{m}' < \mathfrak{m} < B$  gives  $\mathfrak{m} \leq_s \mathfrak{m}'$ ). Likewise, for  $\mathfrak{m} < \mathfrak{m}'' <_s \mathfrak{m}$ , we have  $b \leq \mathfrak{m}''$  for some  $b \in B$ . It follows that for  $\mathfrak{m}'$  as above and  $k \in \mathbb{N}^{\geq 1}$  we have  $\exp(\mathfrak{m}')^k \leq \exp(a)$  for some  $a \in A$ , and that for  $\mathfrak{m}''$  as above and  $k \in \mathbb{N}^{\geq 1}$  we have  $\exp(b) \leq \exp(\mathfrak{m}'')^{1/k}$  for some  $b \in B$ . This yields the desired result in view of [BM, Theorem 3.8 (1)].  $\square$

The monomial representation  $\omega = \mathbb{N} | \emptyset$  shows that in the conclusion of Lemma 8 we cannot drop  $\mathfrak{m}^{\mathbb{N}}$ . Below we use the binary relations  $\prec^L$  and  $\prec^L$  from [BM]. Let  $x = \{x'\} | \{x''\}$  be the canonical representation of  $x$ , and let  $j, k$  range over  $\mathbb{N}^{\geq 1}$ . Then by [BM, Definition 5.12], the defining representation of  $\lambda_x$  is given by

$$\lambda_x = \{k, \exp_j(k \log_j(\lambda_{x'}))\} | \left\{ \exp_j\left(\frac{1}{k} \log_j(\lambda_{x''})\right) \right\}.$$

**Proposition 9.** *We have  $\lambda_{x+1} = \exp(\lambda_x)$ , and thus  $\lambda_{x-1} = \log(\lambda_x)$ .*

**Proof.** Let  $x = \{x'\} | \{x''\}$  be the canonical representation of  $x$ . Then  $1 = 0 | \emptyset$  gives  $x+1 = \{x, x'+1\} | \{x''+1\}$ . Assume inductively that  $\lambda_{x'+1} = \exp(\lambda_{x'})$  and  $\lambda_{x''+1} = \exp(\lambda_{x''})$  for all  $x'$  and  $x''$ . With  $j, k$  ranging over  $\mathbb{N}^{\geq 1}$ , [BM, 5.15] gives

$$\lambda_{x+1} = \{k, \exp_j(k \log_j(\lambda_x)), \exp_j(k \log_j(\lambda_{x'+1}))\} \left| \left\{ \exp_j\left(\frac{1}{k} \log_j(\lambda_{x''+1})\right) \right\} \right. = \{k, \exp_j(k \log_j(\lambda_x)), \exp_j(k \log_{j-1}(\lambda_{x'}))\} | \left\{ \exp_j\left(\frac{1}{k} \log_j(\lambda_{x''})\right) \right\}$$



The defining representation  $\lambda_x = A|B$  is monomial, and the above gives  $\lambda_{x+1} = \mathbb{N} \cup S \cup \exp(A) | \exp(B)$  where  $S$  includes  $\lambda_x^{\mathbb{N}}$  and all elements of  $S$  are  $\asymp^L \lambda_x$ . Since  $\lambda_x \prec^L \exp(\lambda_x)$ , it follows easily from Lemma 8 that  $\lambda_{x+1} = \exp(\lambda_x)$ .  $\square$

**Lemma 10.** *We have  $\lambda_{-\alpha} = \omega^{\omega^{-\alpha}}$ .*

**Proof.** By induction on  $\alpha$ . The case  $\alpha = 0$  holds since  $\lambda_0 = \omega$ . Assuming it holds for a certain  $\alpha$ , we have

$$\lambda_{-(\alpha+1)} = \log \lambda_{-\alpha} = \log \omega^{\omega^{-\alpha}} = \omega^{\omega^{-(\alpha+1)}}.$$

Next, let  $\mu$  be an infinite limit ordinal. Then  $-\mu = \emptyset | \{-\alpha : \alpha < \mu\}$ , and so by [BM, 5.15] and with  $j, k$  ranging over  $\mathbb{N}^{\geq 1}$  we have

$$\lambda_{-\mu} = \mathbb{N} \mid \left\{ \exp_j \left( \frac{1}{k} \log_j \lambda_{-\alpha} \right) \right\}.$$

Now  $\exp_j \left( \frac{1}{k} \log_j \lambda_{-\alpha} \right) \asymp^L \lambda_{-\alpha} \succ^L \lambda_{-\beta}$  when  $\alpha < \beta$ , so by cofinality and the inductive assumption we have

$$\lambda_{-\mu} = \mathbb{N} \mid \{ \omega^{\omega^{-\alpha}} : \alpha < \mu \}.$$

From  $\mathbb{N} < \omega^{\omega^{-\mu}} < \omega^{\omega^{-\alpha}}$  for all  $\alpha < \mu$ , we get  $\lambda_{-\mu} \leq_s \omega^{\omega^{-\mu}}$ . Take  $a$  such that  $\lambda_{-\mu} = \omega^{\omega^{-a}}$ . Then  $\lambda_{-\mu} < \omega^{\omega^{-\alpha}}$  for  $\alpha < \mu$  gives  $\omega^{-a} < \omega^{-\alpha}$  for all  $\alpha < \mu$ , and thus  $a > \alpha$  for all  $\alpha < \mu$ . This yields  $\mu \leq_s a$ , and thus  $\omega^{\omega^{-\mu}} \leq_s \lambda_{-\mu}$ , hence  $a = \mu$ .  $\square$

**Lemma 11.** *For  $\lambda \in \mathfrak{A}$  we have:  $\lambda < \lambda_{-\alpha} \iff \lambda_{-(\alpha+1)} \leq_s \lambda$ .*

**Proof.** For  $\lambda = \lambda_x$  we have the equivalences  
 $\langle \text{pushQED} \mid \square \rangle \lambda_x < \lambda_{-\alpha} \iff x < -\alpha \iff \alpha < -x \iff \alpha + 1 \leq_s -x \iff -(\alpha + 1) \leq_s x \iff \lambda_{-(\alpha+1)} \leq_s \lambda_x. \langle \text{popQED} \rangle \square$   
 $\langle \text{let} \rangle \square$

### Transfinitely iterating the logarithm function.

In view of  $\lambda_{-n} = \log_n \omega$  and the proof of Lemma 10 it is suggestive to think of  $\lambda_{-\alpha}$  as the  $\alpha$  times iterated function  $\log$  evaluated at  $\omega$ . Accordingly we set  $\log_\alpha \omega := \lambda_{-\alpha}$ . We note that for  $\beta < \alpha$  we have  $-\beta < -\alpha$ , so  $\omega^{-\beta} < \omega^{-\alpha}$ , and thus  $\log_\beta \omega < \log_\alpha \omega$ .

**Lemma 12.** *Suppose  $\alpha$  is an infinite limit ordinal. Then  $\log_\alpha \omega$  is the simplest surreal  $x > \mathbb{N}$  such that  $x < \log_\beta \omega$  for all  $\beta < \alpha$ .*

**Proof.** First,  $\mathbb{N} < \log_\alpha \omega < \log_\beta \omega$  for all  $\beta < \alpha$ . Let  $x$  be the simplest surreal  $> \mathbb{N}$  such that  $x < \log_\beta \omega$  for all  $\beta < \alpha$ . Then  $x$  is the simplest positive element in its archimedean class, so  $x = \omega^y$  with  $y > 0$ . Then  $x = \omega^y < \omega^{\omega^{-\beta}}$  for  $\beta < \alpha$  gives  $y < \omega^{-\beta}$  for all  $\beta < \alpha$ . Then  $y$  is the simplest positive element in its archimedean class: if  $0 < y_0 \leq_s y$  and  $y_0 \leq_n y$ , then  $\omega^{y_0} \leq_s \omega^y = x$  and  $\mathbb{N} < \omega^{y_0} \leq x^n < \log_\beta \omega$  for all  $\beta < \alpha$ , so  $\omega^{y_0} = \omega^y$ , and thus  $y_0 = y$ . Hence  $y = \omega^z$  with  $z < -\beta$  for all  $\beta < \alpha$ , and thus  $z \leq -\alpha \leq_s z$ . Therefore,  $\omega^{-\alpha} \leq_s \omega^z = y$ , so

$$\log_\alpha \omega = \omega^{\omega^{-\alpha}} \leq_s \omega^y = x,$$

and thus  $\log_\alpha \omega = x$ .  $\square$

The surreals  $\log_\alpha \omega$  occur in the definition of  $\partial_{\text{BM}}$  later in this section.

### The $\kappa$ -numbers.

The definition of  $\partial_{\text{BM}}$  in [BM] also involves the surreals  $\kappa_x \in \mathfrak{A}$  defined by Kuhlmann and Matusinski [KM]. This is only needed for  $x = -\alpha$ , and it follows from the results in [KM] that  $\kappa_{-\alpha} = \omega^{\omega^{-\omega\alpha}}$ , where  $\omega\alpha$  is the usual ordinal product. Thus in view of Lemma 10:

**Corollary 13.** *We have  $\kappa_{-\alpha} = \lambda_{-\omega\alpha} = \omega^{\omega^{-\omega\alpha}} = \log_{\omega\alpha} \omega$ .*

We also use the binary relations  $\preceq^K$ ,  $\succ^K$ , and  $\asymp^K$  on  $\mathbf{No}^{>\mathbb{N}}$  defined by

$$\begin{aligned} x \preceq^K y &\iff x \leq \exp_n(y) \text{ for some } n, \\ x \succ^K y &\iff x > \exp_n(y) \text{ for all } n, \\ x \asymp^K y &\iff x \preceq^K y \text{ and } y \preceq^K x. \end{aligned}$$

We refer to [BM, 5.3] for proofs of some basic facts about these relations and the  $\kappa_x$  such as:  $\asymp^K$  is an equivalence relation on  $\mathbf{No}^{>\mathbb{N}}$  with convex equivalence classes, every  $\asymp^K$ -equivalence class has a unique element  $\kappa_x$  in it, and this element is the simplest element of this equivalence class. Also,  $\kappa_x \leq_s \kappa_y$  iff  $x \leq_s y$ .

### Defining the pre-derivation for $\partial_{\text{BM}}$ .

The pre-derivation  $D$  with  $\partial_D = \partial_{\text{BM}}$  is denoted by  $\partial_{\mathfrak{L}}$  in [BM, Definition 6.7], and by  $\partial_{\mathfrak{A}}$  in this paper. It is given by

$$\partial_{\mathfrak{A}}(\lambda) := \prod_n \log_n \lambda \Big/ \prod_{\alpha} \log_{\alpha} \omega$$

with  $\alpha$  in the denominator ranging over the ordinals such that  $\log_{\alpha} \omega \geq \log_n \lambda$  for some  $n$ ; to facilitate comparison with [BM] we note that this condition on  $\alpha$  is equivalent to  $\lambda \preceq^K \log_{\alpha} \omega$ . (The products on the right exist, since  $\log_n \lambda$  and  $\log_{\alpha} \omega$  are strictly decreasing as functions of  $n$  and  $\alpha$ , respectively.) The above defining identity for  $\partial_{\mathfrak{A}}$  simplifies the expression in [BM] by our use of infinite products (instead of exponentials of infinite sums), and of Lemma 10 and Corollary 13 (to get rid of  $\kappa$ -numbers). As [BM, Section 9] shows,  $\partial_{\mathfrak{A}}$  is in a certain technical sense the *simplest* pre-derivation.

If  $\lambda > \exp_n \omega$  for all  $n$ , then  $\partial_{\mathfrak{A}}(\lambda) = \prod_n \log_n \lambda$ . Another special case is  $\partial_{\mathfrak{A}}(\log_{\alpha} \omega) = 1 / \prod_{\beta < \alpha} \log_{\beta} \omega$ , in particular,  $\partial_{\mathfrak{A}}(\omega) = 1$ . For  $\epsilon$ -numbers we get the following (not needed later, but included as an example):

**Lemma 14.** *We have  $\log_n \epsilon = \omega^{\omega^{\epsilon-n}}$ . Hence  $\epsilon \in \mathfrak{A}$  and*

$$\partial_{\mathfrak{A}}(\epsilon) = \omega^{\omega^{\epsilon} + \omega^{\epsilon-1} + \omega^{\epsilon-2} + \dots} = \omega^{\epsilon/(1-\omega^{-1})}.$$

**Proof.** From [G, pp. 179, 180] we get that if  $b$ , as a sequence of pluses and minuses, equals  $\epsilon$  followed by  $\epsilon \omega n$  minuses, with  $n \geq 1$  and  $\epsilon \omega n$  being the ordinal product, then  $b = \omega^{\epsilon-n}$ , and  $g(b) = \epsilon - (n-1)$ . In other words,

$$g(\omega^{\epsilon-n}) = \epsilon - (n-1) \quad (n \geq 1).$$

Using this we prove the lemma by induction on  $n$ . The case  $n=0$  is clear. Assume inductively that  $\log_n \epsilon = \omega^{\omega^{\epsilon-n}}$ . Since  $g(\omega^{\epsilon-(n+1)}) = \epsilon - n$ , this gives

$$\exp(\omega^{\omega^{\epsilon-(n+1)}}) = \omega^{\omega^{\epsilon-n}},$$

from which we get  $\log_{n+1} \epsilon = \omega^{\epsilon - (n+1)}$ , as desired.  $\square$

### 3. EXHIBITING $\mathbf{No}$ AS A SUITABLE DIRECTED UNION

At the end of Section 1 we explained how proving  $\mathbb{T} \equiv \mathbf{No}$  (as differential fields) reduces to representing  $\mathbf{No}$  as a directed union of spherically complete grounded  $H$ -subfields. In this section we obtain such a representation. The reader should beware of considering  $\mathbf{No}$  itself as spherically complete, even though the Conway normal form is sometimes summarized as “ $\mathbf{No} = \mathbb{R}(\omega^{\mathbf{No}})$ ”. This is misleading, however, since it suggests that a series like  $\sum_{\alpha} \omega^{-\alpha}$ , where the sum is over all ordinals  $\alpha$ , is a surreal number. It might perhaps be viewed as a surreal number in a strictly larger set-theoretic universe, but not in the one we are (tacitly) working in. A better way of understanding  $\mathbf{No}$  as a valued field is as the directed union  $\bigcup_{\Gamma} \mathbb{R}[[\omega^{\Gamma}]]$  with  $\Gamma$  ranging over the subsets of  $\mathbf{No}$  that underly an additive subgroup of  $\mathbf{No}$ ; for example, any  $\alpha$  gives  $\mathbf{No}(\omega^{\alpha})$  as such a  $\Gamma$ . For any such  $\Gamma$  the corresponding  $\mathbb{R}[[\omega^{\Gamma}]]$  is indeed a spherically complete valued subfield of  $\mathbf{No}$ , but in general  $\mathbb{R}[[\omega^{\Gamma}]]$  is not closed under  $\partial_{\text{BM}}$ , and even if it is, it might not be grounded.

In this section we show that for  $S = \mathbf{No}(\epsilon) \cup \{-\epsilon\}$ , with  $\epsilon$  any  $\epsilon$ -number, the Hahn subgroup  $\Gamma = \mathbb{R}[[\omega^S]]$  of  $\mathbf{No}$  gives rise to a spherically complete valued subfield  $\mathbb{R}[[\omega^{\Gamma}]]$  that is closed under  $\partial_{\text{BM}}$  and grounded as an  $H$ -subfield of  $\mathbf{No}$ .

#### A length bound for $h$ .

This very useful bound is as follows:

**Lemma 15.**  $l(h(y)) \leq \omega^{l(y)+1}$ .

**Proof.** By [G, p. 172] the canonical representation  $y = \{y'\}|\{y''\}$  yields

$$h(y) = \{0, h(y')\}|\{h(y''), \omega^y/2^n\}.$$

We can assume inductively that the lemma holds for the  $y'$  and  $y''$  instead of  $y$ , and thus  $l(h(y')) \leq \omega^{l(y')+1} < \omega^{l(y)+1}$  for all  $y'$ , and likewise with  $y''$  instead of  $y'$ . Also,  $l(\omega^y/2^n) \leq l(\omega^y)l(1/2^n) < \omega^{l(y)}\omega = \omega^{l(y)+1}$ , using [DE, Lemmas 3.6 and 4.1]. Now appeal to [G, Theorem 2.3].  $\square$

Recall from Section 1 that  $h(-\alpha) = \omega^{-(\alpha+1)}$ , and so  $h(0) = \omega^{-1}$  shows that for  $y = 0$  the upper bound in Lemma 15 is attained.

#### Some spherically complete initial subfields of $\mathbf{No}$ .

In this subsection we fix an initial subset  $I$  of  $\mathbf{No}$ . Then  $\Gamma := \mathbb{R}[[\omega^I]]$  is an initial additive subgroup of  $\mathbf{No}$  by the proof of Theorem 18 in [Eh]. (That theorem considers Hahn fields rather than the Hahn group  $\Gamma$ , but the same ideas work; we stress that it is the proof of that theorem rather than its statement that matters here.) Moreover, as Philip Ehrlich mentioned to one of us:

**Lemma 16.** *Suppose  $I$  has a least element  $a$ . Then  $a = -\alpha$  for some  $\alpha$ , and  $\Gamma$  has a least nontrivial archimedean class represented by  $\omega^a$ .*

**Proof.** Taking the longest initial segment of  $a$  consisting of minus signs we get the largest ordinal  $\alpha$  with  $-\alpha \leq_s a$ . Then  $-\alpha \in I$  and  $-\alpha \leq a$ , so  $-\alpha = a$ .  $\square$

Since  $\Gamma$  is initial and an ordered additive group it leads to the initial subfield  $K := \mathbb{R}[[\omega^\Gamma]]$  of  $\mathbf{No}$ . Note that  $K$  is spherically complete, and if  $(a_i)$  is a family in  $K$  for which  $\sum_i a_i$  exists, then  $\sum_i a_i \in K$ . Now  $\Gamma = \mathbb{R}[[\omega^I]]$  is also closed under infinite sums, so if  $(\mathfrak{m}_i)$  is a family in  $\mathfrak{M} \cap K$  such that  $\prod_i \mathfrak{m}_i$  exists, then  $\prod_i \mathfrak{m}_i \in K$ . Thus  $K$  is closed under infinite sums, and also under infinite products of monomials. This is very useful in showing that for suitable choices of  $I$  the field  $K$  is closed under certain surreal derivations. Note however, that if  $I$  has a least element, then  $K^{>\mathbb{N}}$  is not closed under  $\log$ : if  $-\alpha$  is the least element of  $I$ , then  $\log_\alpha \omega = \omega^{\omega^{-\alpha}} \in K$ , but  $\log_{\alpha+1} \omega \notin K$ , as  $-(\alpha+1) \notin I$ .

In order to discuss examples we set  $a^r := \exp(r \log a)$  for  $a > 0$  and  $r \in \mathbb{R}$ , and note agreement with the previously defined  $\omega^r$  when  $a = \omega$ . Moreover,

$$(\log_\alpha \omega)^r = \omega^{r\omega^{-\alpha}} \quad (r \in \mathbb{R}),$$

by the definition of  $a^r$ , using also  $g(\omega^{-(\alpha+1)}) = -\alpha$  and [G, Theorem 10.13].

**Examples 17.** For  $I = \{0\}$  we get  $\Gamma = \mathbb{R}$  and  $K = \mathbb{R}[[\omega^\mathbb{R}]]$ ; note that  $K$  is closed under  $\partial_{\text{BM}}$ , but  $\omega \in K$  and  $\log \omega = \omega^{1/\omega} \notin K$ .

For  $I = \{0, -1\}$  we have  $\Gamma = \mathbb{R} + \mathbb{R}\omega^{-1}$ , so  $\omega^\Gamma = \omega^\mathbb{R}(\log \omega)^\mathbb{R}$ , and thus  $K = \mathbb{R}[[\omega^\mathbb{R}(\log \omega)^\mathbb{R}]]$ , which is again closed under  $\partial_{\text{BM}}$ .

Let  $I = \{\alpha : \alpha \leq \epsilon\}$ . Then  $\epsilon = \omega^\epsilon \in K$ , but Lemma 14 gives  $\log \epsilon \notin K$ , since  $\epsilon - 1 \notin I$  and so  $\omega^{\epsilon-1} \notin \Gamma$ . Likewise we get  $\partial_{\text{BM}}(\epsilon) \notin K$ .

**Lemma 18.** If  $I = \{a : l(a) < \alpha\}$  or  $I = \{a : l(a) \leq \alpha\}$ , then  $I \subseteq \Gamma \subseteq K$ .

**Proof.** Suppose  $I = \{a : l(a) < \alpha\}$ . (The case  $I = \{a : l(a) \leq \alpha\}$  is handled in the same way.) Let  $a \in I$ . Then  $a = \sum_x a_x \omega^x$ , and if  $x \in E(a)$ , then  $l(x) \leq l(\omega^x) \leq l(a) < \alpha$  by [DE, Lemmas 3.4, 4.1, and 4.2], so  $x \in I$ . Thus  $a \in \Gamma$ . This proves  $I \subseteq \Gamma$ . Next, if  $b \in \Gamma$ , then  $b = \sum_{x \in I} b_x \omega^x$ , and so  $b \in K$  in view of  $I \subseteq \Gamma$ .  $\square$

The next lemma will also be crucial:

**Lemma 19.** Suppose  $h(I) \subseteq \Gamma$ . Then  $\log K^> \subseteq K$  and for each  $a \in K$  and term  $t$  of  $a$  we have:  $t$  and all terms of  $\ell(t)$  lie in  $K$ .

**Proof.** Let  $a \in K^>$  have leading monomial  $\mathfrak{m} = \omega^b$  with  $b = \sum_{y \in I} b_y \omega^y$ ; to get  $\log a \in K$ , it is enough that  $\log \mathfrak{m} \in K$ ; the latter holds because  $\log \mathfrak{m} = \sum_y b_y \omega^{h(y)}$ . This proves  $\log K^> \subseteq K$ .

Next, let  $a \in K$  and let  $t$  be a term of  $a$ ; we have to show that  $t$  and all terms of  $\ell(t)$  lie in  $K$ . As  $K \supseteq \mathbb{R}$  is initial, it does contain the term  $t$  of its element  $a$ . We have  $t = r\omega^b$  with  $r \in \mathbb{R}^\times$  and  $b \in \Gamma$ , so  $b = \sum_{y \in I} b_y \omega^y$ , and thus  $\omega^b = \exp(\sum_{y \in I} b_y \omega^{h(y)})$ . Hence  $\ell(t) = \ell(r\omega^b) = \sum_{y \in I} b_y \omega^{h(y)}$  and each of its terms  $b_y \omega^{h(y)}$  lies obviously in  $K$ .  $\square$

**Corollary 20.** If  $h(I) \subseteq \Gamma$  and  $D$  is a pre-derivation with  $D(K \cap \mathfrak{A}) \subseteq K$ , then  $\partial_D(K) \subseteq K$ .

**Proof.** Use the definition of  $\partial_D$  from Section 1, the fact that  $K$  is closed under infinite sums, and Lemma 19.  $\square$

**Corollary 21.** Suppose  $h(I) \subseteq \Gamma$ . Then  $\partial_{\text{BM}}(K) \subseteq K$ .

**Proof.** Let  $\lambda \in K \cap \mathfrak{A}$ ; by Corollary 20 we just need to get  $\partial_{\mathfrak{A}}(\lambda) \in K$ . Since  $K$  is closed under infinite products, it is enough for this to get  $\log_n \lambda \in K$  for all  $n$  (which is the case by Lemma 19), and  $\lambda_{-\alpha} \in K$  for all  $\alpha$  such that  $\lambda \preceq^K \lambda_{-\alpha}$ . Given such  $\alpha$ , take  $n$  with  $\log_n \lambda < \lambda_{-\alpha}$ . Then  $\lambda_{-\alpha} \leq_s \lambda_{-(\alpha+1)} \leq_s \log_n \lambda \in K$  by Lemma 11, and so  $\lambda_{-\alpha} \in K$  because  $K$  is initial.  $\square$

It can happen that  $h(I) \not\subseteq \Gamma$  and that  $K$  is nevertheless closed under  $\partial_{\text{BM}}$ . The next lemma gives a useful criterion for that. To see why that lemma holds, consider a surreal derivation  $\partial$ , and note that from  $\omega^{\omega^y} = \exp(\omega^{h(y)})$  we get

$$\partial(\omega^{\omega^y}) = \omega^{\omega^y} \cdot \partial(\omega^{h(y)}),$$

so for any monomial  $\mathfrak{m} = \omega^b \in K$  we have  $b = \sum_{y \in I} b_y \omega^y$ , and thus

$$\mathfrak{m} = \exp\left(\sum_{y \in I} b_y \omega^{h(y)}\right), \quad \partial(\mathfrak{m}) = \mathfrak{m} \cdot \sum_{y \in I} b_y \partial(\omega^{h(y)}).$$

This leads to:

**Lemma 22.** *Given a surreal derivation  $\partial$ , the following are equivalent:*

1.  $K$  is closed under  $\partial$ ;
2.  $\partial(\omega^{\omega^y}) \in K$  for all  $y \in I$ ;
3.  $\partial(\omega^{h(y)}) \in K$  for all  $y \in I$ .

### The surreal fields $K_\epsilon$ .

Given the  $\epsilon$ -number  $\epsilon$ , we have the initial set  $I := \mathbf{No}(\epsilon)$ , with the corresponding  $\Gamma := \mathbb{R}[[\omega^I]]$  and  $K := \mathbb{R}[[\omega^\Gamma]]$ . In view of Lemmas 15 and 18 we have  $h(I) \subseteq I \subseteq \Gamma$ , so  $\partial_{\text{BM}}(K) \subseteq K$  by Corollary 21. Thus  $K$  is a spherically complete initial  $H$ -subfield of  $\mathbf{No}$ . However,  $I$  has no least element, so  $K$  is not grounded. We repair this by just augmenting  $I$  by  $-\epsilon$ : set  $I_\epsilon := I \cup \{-\epsilon\}$ . Then  $I_\epsilon$  is still initial, with least element  $-\epsilon$ , and so we have the corresponding  $\Gamma_\epsilon := \mathbb{R}[[\omega^{I_\epsilon}]]$  and  $K_\epsilon := \mathbb{R}[[\omega^{\Gamma_\epsilon}]]$ . To get  $\partial_{\text{BM}}(K_\epsilon) \subseteq K_\epsilon$  we note that  $K \subseteq K_\epsilon$ , and so it suffices by Lemma 22 that  $\partial_{\mathfrak{A}}(\omega^{\omega^{-\epsilon}}) \in K_\epsilon$ . But  $\omega^{\omega^{-\epsilon}} = \log_\epsilon \omega$ , and

$$\partial_{\mathfrak{A}}(\log_\epsilon \omega) = 1 \Big/ \prod_{\alpha < \epsilon} \log_\alpha \omega,$$

which lies in  $K$ , and hence in  $K_\epsilon$ . Thus  $K_\epsilon$  is a grounded  $H$ -subfield of  $\mathbf{No}$ , and

$$\mathbf{No} = \bigcup_{\epsilon} K_\epsilon.$$

Note that Corollary 21 does not apply to  $I_\epsilon$ , since  $h(-\epsilon) = \omega^{-(\epsilon+1)} \notin \Gamma$ ; this is why we did the less direct construction via  $I = \mathbf{No}(\epsilon)$ .

Since  $\omega^{-\epsilon}$  represents the smallest archimedean class of  $\Gamma_\epsilon$ , we have

$$\max \Psi_{K_\epsilon} = v((\omega^{\omega^{-\epsilon}})^\dagger) = v((\log_\epsilon \omega)^\dagger)$$

by the proof of Lemma 4. In view of  $(\log_\epsilon \omega)^\dagger = (\log_{\epsilon+1} \omega)'$  and the remarks at the end of Section 1, the representation of  $\mathbf{No}$  as an increasing union  $\bigcup_{\epsilon} K_\epsilon$  of spherically complete grounded  $H$ -subfields now gives  $\partial_{\text{BM}}(\mathbf{No}) = \mathbf{No}$ . (The proof of  $\partial_{\text{BM}}(\mathbf{No}) = \mathbf{No}$  in [BM, Section 7] is different.) Thus by the results stated at the end of Section 1 we conclude that  $\mathbf{No} \equiv \mathbb{T}$ , as differential fields.

## 4. THE CASE OF RESTRICTED LENGTH

A set  $S \subseteq \mathbf{No}$  is said to be of *countable type* if  $l(a)$  is countable for all  $a \in S$ , and all well-ordered subsets of  $S$  as well as all reverse well-ordered subsets of  $S$  are countable. (Note that  $l(a)$  is countable for every  $a \in \mathbf{No}(\omega_1)$ , but that  $\mathbf{No}(\omega_1)$  is not of countable type, since it has the set of countable ordinals as an uncountable well-ordered subset.)

**Proposition 23.** *Suppose the subset  $S$  of  $\mathbf{No}$  is of countable type. Then the additive subgroup  $\mathbb{R}[[\omega^S]]$  of  $\mathbf{No}$  is also of countable type.*

**Proof.** The case  $\alpha = 1$  of Esterle [E, Lemme 2.2] and the remarks following it yield that every well-ordered subset of  $\mathbb{R}[[\omega^S]]$  is countable. Hence every reverse well-ordered subset of  $\mathbb{R}[[\omega^S]]$  is countable as well. Let  $a \in \mathbb{R}[[\omega^S]]$ . Then  $a = \sum_{s \in E(a)} a_s \omega^s$ . Now  $E(a) \subseteq S$  is countable, so the well-ordered set  $-E(a)$  has order type  $\mu < \omega_1$ . Since  $\omega_1$  is regular, we have a countable ordinal  $\nu$  such that  $l(s) \leq \nu$  for all  $s \in E(a)$ . Then  $l(\omega^s) \leq \omega^\nu$  for all  $s \in E(a)$  by [DE, Lemma 4.1], hence  $l(a_s \omega^s) \leq \omega^{\nu+1}$  for all  $s \in E(a)$  by [DE, Proposition 3.6]. Thus

$$l(a) \leq \mu \cdot \omega^{\nu+1} < \omega_1,$$

by [G, Theorem 5.12], or [DE, Lemma 4.2, (3)].  $\square$

As an example, consider  $S := \mathbf{No}(\omega)$ , the set of dyadic numbers. Then  $S$  is of countable type, and so  $\mathbb{R}[[\omega^S]]$  is of countable type. Nevertheless,  $l(\mathbb{R}[[\omega^S]])$  is cofinal in  $\omega_1$ : given any countable ordinal  $\mu$ , take an order reversing injective map  $\alpha \mapsto s_\alpha: \mu \rightarrow S$ ; then  $a := \sum_\alpha \omega^{s_\alpha} \in \mathbb{R}[[\omega^S]]$  has  $l(a) \geq \mu$ , by [G, p. 63].

Let  $\kappa$  be any infinite cardinal. Esterle [E, Lemme 2.2] actually tells us for any set  $S \subseteq \mathbf{No}$ : if all well-ordered subsets and all reverse well-ordered subsets of  $S$  have size  $\leq \kappa$ , then this remains true for the set  $\mathbb{R}[[\omega^S]] \subseteq \mathbf{No}$ . The next cardinal  $\kappa^+$  is regular, so the arguments in the proof of Proposition 23 go through to give the following, where we call  $S \subseteq \mathbf{No}$  of *type  $\kappa$*  if  $l(a) \leq \kappa$  for all  $a \in S$  and all well-ordered subsets of  $S$  and all reverse well-ordered subsets of  $S$  have size  $\leq \kappa$ .

**Corollary 24.** *If  $S \subseteq \mathbf{No}$  is of type  $\kappa$ , then so is  $\mathbb{R}[[\omega^S]]$ .*

Next we show that for countable  $\mu$  the set  $\mathbf{No}(\mu)$  is of countable type. Every element of  $\mathbf{No}(\mu)$  has clearly countable length, for countable  $\mu$ , and  $\mathbf{No}(\mu)$  is closed under  $x \mapsto -x$ , so the assertion above reduces to:

**Lemma 25.** *Suppose the ordinal  $\mu$  is countable. Then every well-ordered subset of  $\mathbf{No}(\mu)$  is countable.*

This may remind the reader of the well-known property of the ordered set  $\mathbb{R}$  that every well-ordered subset of  $\mathbb{R}$  is countable. Here is a quick proof using that  $\mathbb{R}$  has a countable dense subset  $\mathbb{Q}$ : given any embedding  $\alpha \mapsto r_\alpha$  of an infinite cardinal  $\kappa$  into  $\mathbb{R}$ , pick for each  $\alpha < \kappa$  a rational  $q_\alpha$  such that  $r_\alpha < q_\alpha < r_{\alpha+1}$ ; it follows that  $\kappa = \aleph_0$ . However, such a countable density argument cannot be used for ordered sets  $\mathbf{No}(\mu)$  when  $\mu$  is a countable limit ordinal  $> \omega$ :

**Lemma 26.** *Let  $\mu$  be an infinite limit ordinal. Then the ordered set  $\mathbf{No}(\mu)$  is dense without endpoints. If  $\mu > \omega$ , then there exists a collection of  $2^{\aleph_0}$  pairwise disjoint open intervals in  $\mathbf{No}(\mu)$ , which has therefore no countable dense subset.*

**Proof.** The ordinals  $\alpha < \mu$  are cofinal in this ordered set, and there is no largest such  $\alpha$ . For  $a < b$  in this ordered set, take  $\alpha \leq l(a), l(b)$  such that  $a|_\alpha = b|_\alpha$  and  $a(\alpha) < b(\alpha)$ . If  $l(b) > \alpha$ , then  $b(\alpha) = +$ , so  $a < b - < b$ . If  $l(a) > \alpha$ , then  $a(\alpha) = -$ , so  $a < a + < b$ . Note that  $b -, a + \in \mathbf{No}(\mu)$ , as  $\mu$  is a limit ordinal,

Next, assume  $\mu > \omega$ . For each nondyadic  $r \in \mathbb{R} \subseteq \mathbf{No}$ , we have the surreals  $r -$  and  $r +$  of length  $\omega + 1$ , and so we obtain the pairwise disjoint open intervals  $(r -, r +)$  in  $\mathbf{No}(\mu)$ .  $\square$

**Proof. (Proof of Lemma 25)** For  $a \in \mathbf{No}(\mu)$  we define  $\hat{a}: \mu \rightarrow \mathbb{R}$  by

$$\hat{a}(\alpha) = \begin{cases} -1 & \text{if } a(\alpha) = -, \\ 0 & \text{if } a(\alpha) = 0, \\ 1 & \text{if } a(\alpha) = +, \end{cases}$$

For  $S = \{\alpha: \alpha < \mu\}$  this yields an order-preserving injective map

$$a \mapsto \sum_{\alpha < \mu} \hat{a}(\alpha) \omega^{-\alpha}: \mathbf{No}(\mu) \rightarrow \mathbb{R}[[\omega^S]].$$

It remains to appeal to Proposition 23.  $\square$

Essentially the same argument yields the following generalization:

**Corollary 27.** *If  $\kappa$  is an infinite cardinal and  $\mu$  is an ordinal of cardinality  $\leq \kappa$ , then each well-ordered subset of  $\mathbf{No}(\mu)$  has cardinality  $\leq \kappa$ .*

Note that for a countable  $\epsilon$ -number  $\epsilon$  the initial set  $I_\epsilon = \mathbf{No}(\epsilon) \cup \{-\epsilon\}$  is of countable type by Lemma 25, and hence  $\Gamma_\epsilon$  and  $K_\epsilon$  are as well by Proposition 23. Taking the union over all such countable  $\epsilon$  we obtain the set  $\mathbf{No}(\omega_1)$  of all surreals of countable length as an increasing union of spherically complete grounded  $H$ -subfields  $K_\epsilon$  of  $\mathbf{No}$ . As in Section 3 and using also the model completeness of  $T_{\text{small}}^{\text{nl}} = \text{Th}(\mathbb{T})$  this yields Theorem 2. The results above lead moreover to the following generalization:

**Corollary 28.** *Let  $\kappa$  be any uncountable cardinal. Then the subfield  $\mathbf{No}(\kappa)$  of  $\mathbf{No}$  is closed under  $\partial_{\text{BM}}$ , and  $\mathbf{No}(\kappa) \prec \mathbf{No}$ , as ordered differential fields.*

**Proof.** If  $\kappa$  is regular we can argue as for  $\omega_1$ , using Corollaries 24 and 27 instead of Proposition 23 and Lemma 25. If  $\kappa$  is singular, use that it is the supremum of the uncountable regular cardinals below it.  $\square$

## 5. CONSTRUCTING EMBEDDINGS

So far we have just worked inside  $\mathbf{No}$  and established Theorem 2. In this section we turn to  $\mathbb{T}$  and prove the embedding results: Theorems 1 and 3.

### Embedding $\mathbb{T}$ into $\mathbf{No}$ .

Given a Hahn field  $\mathbb{R}[[G]]$  over  $\mathbb{R}$  we define a map  $F: \mathbb{R}[[G]] \rightarrow \mathbf{No}$  to be *strongly additive* if for every summable family  $(f_i)$  in  $\mathbb{R}[[G]]$  the family  $(F(f_i))$  is summable in  $\mathbf{No}$  and  $F(\sum_i f_i) = \sum_i F(f_i)$ . We refer to [ADH, Appendix A] for the construction of  $\mathbb{T}$  as an exponential ordered field. In this construction  $\mathbb{T}$  is a subfield of a Hahn field  $\mathbb{R}[[G^{\text{LE}}]]$ : in fact,  $G^{\text{LE}}$  is a certain directed union of ordered subgroups  $G_m \downarrow_n$ , and  $\mathbb{T}$  is the corresponding directed union of the Hahn fields  $\mathbb{R}[[G_m \downarrow_n]]$ . A map  $F: \mathbb{T} \rightarrow \mathbf{No}$  is said to be *strongly additive* if its restriction to each  $\mathbb{R}[[G_m \downarrow_n]]$  is strongly additive.

**Proposition 29.** *There is a unique strongly additive embedding  $\iota: \mathbb{T} \rightarrow \mathbf{No}$  of exponential ordered fields that is the identity on  $\mathbb{R}$  and such that  $\iota(x_{\mathbb{T}}) = \omega$ .*



**Proof.** We use the notations from [ADH, Appendix A] except that the  $x$  there is  $x_{\mathbb{T}}$  here. The construction of  $\mathbb{T}$  there begins with the Hahn field  $E_0 = \mathbb{R}[[x_{\mathbb{T}}^{\mathbb{R}}]]$ , and there is clearly a (unique) strongly additive ordered field embedding  $i_0: E_0 \rightarrow \mathbf{No}$  such that  $i_0(r) = r$  and  $i_0(x_{\mathbb{T}}^r) = \omega^r$  for all  $r \in \mathbb{R}$ . Moreover,  $i_0(\langle \text{ex} \rangle^b) = \exp(i_0(b))$  for all  $b \in B_0$ , and  $\exp(i_0(a)) > i_0(E_0)$  for all  $a \in A_0^>$ . Assume inductively that we have an extension of  $i_0$  to a strongly additive ordered field embedding  $i_m: E_m = \mathbb{R}[[G_m]] \rightarrow \mathbf{No}$  such that  $i_m(\langle \text{ex} \rangle^b) = \exp(i_m(b))$  for all  $b \in B_m$ , and  $\exp(i_m(a)) > i_m(E_m)$  for all  $a \in A_m^>$ . Then one checks easily that  $i_m$  extends (uniquely) to a strongly additive ordered field embedding  $i_{m+1}: E_{m+1} \rightarrow \mathbf{No}$  such that  $i_{m+1}(\langle \text{ex} \rangle^b) = \exp(i_{m+1}(b))$  for all  $b \in B_{m+1}$ , and  $\exp(i_{m+1}(a)) > i_{m+1}(E_{m+1})$  for all  $a \in A_{m+1}^>$ . Taking a union over all  $m$  we obtain an embedding

$$\iota_0 := \bigcup_m i_m: \mathbb{R}[[x_{\mathbb{T}}^{\mathbb{R}}]]^{\mathbb{E}} = \bigcup_m \mathbb{R}[[G_m]] \rightarrow \mathbf{No}$$

of ordered exponential fields. Replacing in the above  $\ell_0 = x_{\mathbb{T}}$ ,  $G_m$ ,  $\omega$ , by  $\ell_n = \log_n x_{\mathbb{T}}$ ,  $G_m \downarrow_n$ ,  $\log_n \omega$ , respectively, we obtain likewise an embedding

$$\iota_n: \mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathbb{E}} = \bigcup_m \mathbb{R}[[G_m \downarrow_n]] \rightarrow \mathbf{No}$$

of ordered exponential fields with  $\iota_n(\ell_n) = \log_n \omega$ . Each  $\iota_{n+1}$  extends  $\iota_n$ , so we can take the union over all  $n$  to get an embedding  $\iota: \mathbb{T} \rightarrow \mathbf{No}$  as claimed. The uniqueness holds because the smallest subfield of  $\mathbb{T}$  that contains  $\mathbb{R}(x_{\mathbb{T}})$  and is closed under exponentiation, taking logarithms of positive elements, and summation of summable families is  $\mathbb{T}$  itself.  $\square$

Next we apply the model completeness of the theory of the exponential ordered field of real numbers (Wilkie [W]). By [DMM] and [DE], respectively, the ordered exponential fields  $\mathbb{T}$  and  $\mathbf{No}$  are models of this theory, and so  $\iota: \mathbb{T} \rightarrow \mathbf{No}$  is an elementary embedding of ordered exponential fields.

It is easy to check that  $\iota: \mathbb{T} \rightarrow \mathbf{No}$  is also an embedding of ordered differential fields. In view of  $\mathbb{T} \equiv \mathbf{No}$  (as differential fields), and the model completeness of  $T_{\text{small}}^{\text{nl}}$  mentioned at the end of Section 1 we conclude that  $\iota$  is an elementary embedding of ordered differential fields: Theorem 1.

Is  $\iota$  an elementary embedding of *ordered differential exponential fields*? We don't know; this is related to the open problem from [ADH] to extend the model-theoretic results there about  $\mathbb{T}$  as a differential field to  $\mathbb{T}$  as a differential exponential field.

It follows easily from the construction of  $\mathbb{T}$  and  $\iota$  that all surreal derivations  $\partial$  with  $\partial(\omega) = 1$  agree on  $\iota(\mathbb{T})$ .

**Proposition 30.** *Here are some further properties of the map  $\iota$ :*

1.  $\iota(G^{\text{LE}}) = \mathfrak{M} \cap \iota(\mathbb{T})$ ;
2.  $\iota(\mathbb{T})$  is truncation closed;
3.  $\iota(\mathbb{T})$  is of countable type; in particular,  $\iota(\mathbb{T}) \subseteq \mathbf{No}(\omega_1)$ .

**Proof.** Induction on  $m$  gives  $\iota(G_m) \subseteq \mathfrak{M}$ , where we use at the inductive step that  $G_{m+1} = \exp(A_m) G_m$  and  $\iota(A_m) \subseteq \mathbb{J}$ , the latter being a consequence of  $\iota(G_m) \subseteq \mathfrak{M}$ . Likewise,  $\iota(G_m \downarrow_n) \subseteq \mathfrak{M}$  for all  $m, n$ , and thus  $\iota(G^{\text{LE}}) \subseteq \mathfrak{M}$ . Since  $\iota$  respects infinite sums of monomials, this yields (1), and (2) is then an immediate consequence using also that  $\mathbb{T}$  is truncation closed in  $\mathbb{R}[[G^{\text{LE}}]]$ . As to (3), using the results in Section 4 one shows by induction on  $m$  that  $\iota(G_m)$ , and likewise each  $\iota(G_m \downarrow_n)$ , has countable type. Hence  $\iota(G^{\text{LE}})$  has countable type, and so does  $\iota(\mathbb{T})$ .  $\square$



*Question* (Elliot Kaplan): can (2) be improved to  $\iota(\mathbb{T})$  being initial?

### Embedding $H$ -fields into $\mathbf{No}$ .

Let  $\epsilon$  be an  $\epsilon$ -number; for example,  $\epsilon$  could be any uncountable cardinal. We recall from [DE] that  $\mathbf{No}(\epsilon)$  is a real closed subfield of  $\mathbf{No}$  containing  $\mathbb{R}$ . We consider  $\mathbf{No}(\epsilon)$  as a valued subfield of  $\mathbf{No}$  with (divisible) ordered value group  $v(\mathbf{No}(\epsilon)^\times)$ . We shall need an easy auxiliary result:

**Lemma 31.** *Let  $\kappa$  be a regular uncountable cardinal. Then the underlying ordered sets of  $\mathbf{No}(\kappa)$  and  $v(\mathbf{No}(\kappa)^\times)$  are  $\kappa$ -saturated.*

**Proof.** Let  $A, B \subseteq \mathbf{No}(\kappa)$  have cardinality  $< \kappa$ , with  $A < B$ . The regularity of  $\kappa$  yields an ordinal  $\alpha < \kappa$  such that  $l(A \cup B) < \alpha$ . By [G, Theorem 2.3] this gives a surreal  $a$  with  $l(a) \leq \alpha$  such that  $A < a < B$ , and then  $a \in \mathbf{No}(\kappa)$ . Thus  $\mathbf{No}(\kappa)$  is  $\kappa$ -saturated as an ordered set. Next, let  $P, Q \subseteq \mathbf{No}(\kappa)^\times$  have cardinality  $< \kappa$ , with  $v(P) > v(Q)$ . Set  $A := \{np : n \geq 1, p \in P\}$  and  $B := \{q/n : n \geq 1, q \in Q\}$ . Then  $A < B$ , and so the above gives  $a \in \mathbf{No}(\kappa)$  with  $A < a < B$ . Then  $v(P) > v(a) > v(Q)$ , showing that  $v(\mathbf{No}(\kappa)^\times)$  is  $\kappa$ -saturated as an ordered set.  $\square$

For Theorem 3 we need a sharpening of the model completeness of the theory  $T^{\text{nl}}$  of  $\omega$ -free newtonian Liouville closed  $H$ -fields, namely, the quantifier elimination (QE) explained in [ADH, Introduction to Chapter 16]. The relevant first-order language for QE has in addition to  $\mathcal{L}$  extra unary predicate symbols  $I, \Lambda, \Omega$ , to be interpreted in a model  $L$  of  $T^{\text{nl}}$  as sets  $I(L), \Lambda(L), \Omega(L) \subseteq L$  according to their defining axioms:

$$\begin{aligned} I(a) &\iff a = y' \text{ for some } y \prec 1 \text{ in } L, \\ \Lambda(a) &\iff a = -y^{\dagger\dagger} \text{ for some } y \succ 1 \text{ in } L, \\ \Omega(a) &\iff 4y'' + ay = 0 \text{ for some } y \in L^\times. \end{aligned}$$

The sets  $I(L), \Lambda(L), \Omega(L) \subseteq L$  are convex; their role with respect to QE is like that of the set of squares in a real closed field. For more on this, see [ADH, Introduction]. A  $\Lambda\Omega$ -field is a substructure  $\mathbf{K} = (K, I, \Lambda, \Omega)$  of such an expanded model  $(L, \dots)$  of  $T^{\text{nl}}$  for which  $K$  is an  $H$ -subfield of  $L$ . This notion of a  $\Lambda\Omega$ -field is studied in detail in [ADH, Section 16.3], from which we take in particular the fact that any  $\omega$ -free  $H$ -field  $K$  has a unique expansion to a  $\Lambda\Omega$ -field  $\mathbf{K} = (K, I, \Lambda, \Omega)$ . The proof below assumes familiarity with several other results from [ADH, Section 16.3].

**Proof. (Proof of Theorem 3)** Let  $\mathbf{No}_{\Lambda\Omega}$  be the expansion of  $\mathbf{No}$  to a  $\Lambda\Omega$ -field, and let  $K$  be any  $H$ -field with small derivation and constant field  $\mathbb{R}$ . In order to embed  $K$  over  $\mathbb{R}$  into  $\mathbf{No}$ , we first expand  $K$  to a  $\Lambda\Omega$ -field  $\mathbf{K} = (K, I, \Lambda, \Omega)$  with  $1 \notin I$ ; this can be done in at least one way, and at most two ways, and  $1 \notin I$  guarantees that all  $\Lambda\Omega$ -field extensions of  $\mathbf{K}$  have small derivation. We claim that  $\mathbf{K}$  can be embedded into  $\mathbf{No}_{\Lambda\Omega}$ . The ordered field  $\mathbb{R}$  with the trivial derivation is an  $H$ -field and expands to the  $\Lambda\Omega$ -field  $\mathbf{R} := (\mathbb{R}, \{0\}, (-\infty, 0], (-\infty, 0])$ . The inclusion of  $\mathbb{R}$  into  $K$  and into  $\mathbf{No}$  are embeddings of  $\mathbf{R}$  into  $\mathbf{K}$  and  $\mathbf{No}_{\Lambda\Omega}$ , respectively. By taking  $\mathbf{E} := \mathbf{R}$ , our claim reduces therefore to proving the following more general statement:

*Claim.* Let  $\mathbf{E} \subseteq \mathbf{K}$  be an extension of  $\Lambda\Omega$ -fields with  $\mathbb{R}$  as their common constant field, and let  $i: \mathbf{E} \rightarrow \mathbf{No}_{\Lambda\Omega}$  be an embedding of  $\Lambda\Omega$ -fields that is the identity on  $\mathbb{R}$ . Then  $i$  extends to an embedding  $\mathbf{K} \rightarrow \mathbf{No}_{\Lambda\Omega}$  of  $\Lambda\Omega$ -fields.

To prove this we first extend  $\mathbf{K}$  to make it  $\omega$ -free, newtonian, and Liouville closed; by [ADH, 16.4.1 and 14.5.10] this can be done without changing its constant field. Next we apply [ADH, 16.4.1] again, but this time to  $\mathbf{E}$ , to arrange that  $\mathbf{E}$  is  $\omega$ -free. Take a regular uncountable cardinal  $\kappa > \text{card}(K)$  such that  $i(E) \subseteq \mathbf{No}(\kappa)$ , where  $E$  is the underlying set of  $\mathbf{E}$ . By Corollary 28 we have  $\mathbf{No}(\kappa) \prec \mathbf{No}$ . In view of Lemma 31 and [ADH, 16.2.3] we can then extend  $i$  to an embedding  $K \rightarrow \mathbf{No}(\kappa)$ .  $\square$

### Final remarks.

Suppose the  $H$ -field  $K$  has small derivation and constant field  $\mathbb{R}$ . Then Theorem 3 yields an embedding  $i: K \rightarrow \mathbf{No}$  over  $\mathbb{R}$ . Under some reasonable further conditions, like  $K$  being  $\omega$ -free and newtonian, can we take  $i$  such that  $i(K)$  is truncation closed, or even initial? The interest of such a result would depend on how canonical the derivation  $\partial_{\text{BM}}$  is deemed to be. As already mentioned at the end of the introduction, we doubt that  $\partial_{\text{BM}}$  is optimal: the condition on pre-derivations to take values in  $\mathbb{R}^{>\mathfrak{M}}$  seems too narrow. But even with this restriction one can construct pre-derivations  $D \neq \partial$  such that Theorems 1 and 3 go through for  $\mathbf{No}$  equipped with  $\partial_D$  instead of with  $\partial_{\text{BM}}$ , with only minor changes in the proofs.

### BIBLIOGRAPHY

- [AD] M. Aschenbrenner, L. van den Dries, *H-fields and their Liouville extensions*, Math. Z. **242** (2002), 543–588.
- [ADH] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Asymptotic Differential Algebra and Model Theory of Transseries*, arXiv:1509.02588, Ann. of Math. Stud., to appear.
- [BM] A. Berarducci, V. Mantova, *Surreal numbers, derivations, and transseries*, J. Eur. Math. Soc. (JEMS), to appear.
- [C] J. Conway, *On Numbers and Games*, London Mathematical Society Monographs, vol. 6, Academic Press, London, 1976.
- [DE] L. van den Dries and P. Ehrlich, *Fields of surreal numbers and exponentiation*, Fund. Math. **167** (2001), 173–188, and *Erratum*, Fund. Math. **168** (2001), 295–297.
- [DMM] L. van den Dries, A. Macintyre, and D. Marker, *Logarithmic-exponential power series*, J. London Math. Soc. **56** (1997), 417–434.
- [Eh] P. Ehrlich, *Number systems with simplicity hierarchies: a generalization of Conway’s theory of surreal numbers*, J. Symbolic Logic **66**, (2001), 1231–1258.
- [E] J. Esterle, *Solution d’un problème d’Erdős, Gillman et Henriksen et application à l’étude des homomorphismes de  $C(K)$* , Acta Math. (Hungarica) **30** (1977), 113–127.
- [G] H. Gonshor, *An Introduction to the Theory of Surreal Numbers*, London Mathematical Society Lecture Note Series, vol. 110, Cambridge University Press, Cambridge, 1986.
- [vdH:phd] J. van der Hoeven, *Asymptotique Automatique*, Ph. D. thesis, École Polytechnique, 1997.
- [KM] S. Kuhlmann and M. Matusinski, *The exponential-logarithmic equivalence classes of surreal numbers*, Order **32** (2015), 53–68.
- [S] M. C. Schmeling, *Corps de Transséries*, Ph. D. thesis, Université Paris-VII, 2001.
- [W] A. Wilkie, *Some model completeness results for expansions of the ordered field of real numbers by Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), 1051–1094.